Solving the BS PDE the Right Way

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I'd like to give an alternative derivation of the Black-Scholes (BS) PDE not involving the clever (mystifying?) transformation to the heat equation and thus present a more general technique for solving constant coefficeint advection-diffusion PDEs. All we need is the Fourier transform:

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} e^{-i\omega y} f(y) dy,$$

where $f : \mathbb{R} \to \mathbb{R}$ and $f \in L^2$.

We'll use the following well-known facts of the Fourier transform:

1. $\mathcal{F}\left[\frac{1}{s\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-m}{s}\right)^2\right)\right] = \exp(-i\omega m - s^2\omega^2/2)$,

2. $\mathcal{F}\left[\frac{\partial^n f}{\partial y^n}\right] = (i\omega)^n \mathcal{F}[f]$,

3. $\mathcal{F}[cf] = c\mathcal{F}[f]$,

4. $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$,

where the convolution $(f * g)(y) = \int_{-\infty}^{\infty} f(z)g(y-z)dz$.

Here's the BS PDE, stated without boundary or terminal conditions:

$$\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

Step 1 Transform the PDE from forward in time to backward in time, which makes it well-posed. This is done by changing variables

$$t \mapsto T - t =: \tau,$$

which which only affects the t-derivative term in that

$$\frac{\partial C}{\partial t} \mapsto -\frac{\partial C}{\partial \tau}.$$

Thus the forward-time PDE is

$$\frac{\partial C}{\partial \tau} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC. \tag{1}$$

Step 2 Transform the PDE from variable coefficient to constant coefficient. Starting with the PDE backward in time, make the change of variables

$$S \mapsto \log S := x,$$

which results in the derivatives

$$\begin{aligned}\frac{\partial C}{\partial S} &= \frac{\partial C}{\partial x}\frac{1}{S},\\ \frac{\partial^2 C}{\partial S^2} &= \frac{1}{S^2}\left(\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}\right).\end{aligned}$$

Plugging these into (1) we get

$$\begin{aligned}\frac{\partial C}{\partial \tau} &= \frac{\sigma^2 S^2}{2}\frac{1}{S^2}\left(\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}\right) + rS\left(\frac{\partial C}{\partial x}\frac{1}{S}\right) - rC\\ &= \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial C}{\partial x} - rC.\end{aligned}$$

Step 3 Take the Fourier transform of each term term above and solve the resulting separable ODE:

$$\begin{aligned}\frac{\partial \hat{C}}{\partial \tau} &= -\frac{\sigma^2\omega^2}{2}\hat{C} + i\omega\left(r - \frac{\sigma^2}{2}\right)\hat{C} - r\hat{C},\\ \hat{C} &= \hat{C}_0 e^{-r\tau}\exp\left(-\frac{\sigma^2\omega^2}{2}\tau + i\omega\left(r - \frac{\sigma^2}{2}\right)\tau\right).\end{aligned}$$

Step 4 Letting $m = \left(\frac{\sigma^2}{2} - r\right)\tau$ and $s = \sigma\sqrt{\tau}$ from the Fourier transform notation, note

$$\exp\left(-\frac{\sigma^2\omega^2}{2}\tau + i\omega\left(r - \frac{\sigma^2}{2}\right)\tau\right) = \mathcal{F}\left[\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}\left(\frac{x - \left(\frac{\sigma^2}{2} - r\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right)\right],$$

so

$$\begin{aligned}\hat{C} &= \hat{C}_0 e^{-r\tau}\mathcal{F}\left[\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}\left(\frac{x - \left(\frac{\sigma^2}{2} - r\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right)\right]\\ &= \frac{1}{\sigma\sqrt{2\pi\tau}}e^{-r\tau}\mathcal{F}\left[C_0 * \exp\left(-\frac{1}{2}\left(\frac{x - \left(\frac{\sigma^2}{2} - r\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right)\right].\end{aligned}$$

Step 5 Take inverse transform:

$$C(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-r\tau} \int_{-\infty}^{\infty} C_0(z) \exp\left(-\frac{1}{2} \left(\frac{x - z - \left(\frac{\sigma^2}{2} - r\right)\tau}{\sigma\sqrt{\tau}} \right)^2 \right) dz$$

$$C(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-r\tau} \int_{-\infty}^{\infty} C_0(z) \exp\left(-\frac{1}{2} \left(\frac{z - \left(x + \left(r - \frac{\sigma^2}{2}\right)\tau\right)}{\sigma\sqrt{\tau}} \right)^2 \right) dz$$

Step 6 Finally, change variables back $x \to S$, where we had $x = \log S$. Before we do this, note S is really the "initial" stock price in the usual sense, i.e. $S = S_0$, but to be consistent we'll stick with S as the initial (known) stock price. We'll also transform the z variable, suggestively calling it S_T by $S_T = e^z$.

$$C(S,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-r\tau} \int_{0}^{\infty} C_0(S_T) \frac{1}{S_T} \exp\left(-\frac{1}{2} \left(\frac{\log S_T - \left(\log S + \left(r - \frac{\sigma^2}{2}\right)\tau\right)}{\sigma\sqrt{\tau}} \right)^2 \right) dS_T.$$

Just note

$$f(S_T) := \frac{1}{S_T\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{1}{2} \left(\frac{\log S_T - \left(\log S + \left(r - \frac{\sigma^2}{2}\right)\tau\right)}{\sigma\sqrt{\tau}} \right)^2 \right)$$

is the probability density function for a $\log \mathcal{N}\left(\log S + \left(r - \frac{\sigma^2}{2}\right)\tau, \sigma^2\tau\right)$ random variable, and under Black-Scholes, this is indeed the distribution of S_T under $\mathbb{Q}$. Hence

$$C(S,\tau) = e^{-r\tau}\mathbb{E}_{\mathbb{Q}}[C_0(S_T)|\mathcal{F}_t].$$